Lecture 1 - Feb 3

(1) Recall: For a smoothly bdd domain $\Omega \subset \mathbb{C}^n$, and $p \in \partial\Omega =: M$, ($\dim_{\mathbb{R}} M = 2n-1$).

$$T_p^{0,1}M = \left\{ X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial \bar z_j} : X \in \mathbb{C} \otimes T_p M \right\}$$

I.e. if $M = \{\rho = 0\}$, where $d\rho \neq 0$ on M then $X \in T_p^{0,1}M \Leftrightarrow$

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial \bar z_j}(p) X^j = 0.$$

Since $d\rho \neq 0 \Rightarrow \bar\partial \rho := \sum_{j=1}^{n} \frac{\partial \rho}{\partial \bar z_j} d\bar z_j \neq 0$

($d\rho = \partial\rho + \bar\partial\rho = 2\operatorname{Re}\bar\partial\rho$). Thus, linear algebra tells us that $T_p^{0,1}M$ has constant dimension $n-1$ for all $p \in M$. Since $\bar\partial\rho$ varies smoothly, we conclude that $T_p^{0,1}M$ form a rank $n-1$ vector bundle over M.

Moreover, it is clear that if we set $T_p^{1,0}M$ to be the $(1,0)$-vectors (i.e. $\sum X^j \frac{\partial}{\partial z_j} \in \mathbb{C} \otimes T_pM$), then $T^{0,1}M = \overline{T^{1,0}M}$ and their intersection is empty.

The next observation is that if we take two sections (vector fields), X, Y of, say, $T^{0,1}M$, then their commutator $[X,Y] = XY - YX$ is still tangent to M (general property of commutators of vector fields) and it is also clear that $[X,Y]$ is still a $(0,1)$ vector field.

These properties can be introduced on an abstract manifold with a distinguished subbundle of $\mathbb{C} \otimes TM$.

Def. A smooth (real) manifold M along with a distinguished subbundle $\mathcal{V} \subseteq \mathbb{C} \otimes TM$ that satisfy these two properties

(a) $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$

(b) (Formal integrability) For two sections X, Y of $\mathcal{V}$, it holds that $[X, Y]$ is a section of $\mathcal{V}$,

is called a CR manifold

(2) We also saw real submanifolds M of higher codimension in $\mathbb{C}^n$ can be CR, but need not be. The problem is that the dimension of $T^{0,1}_p M$ can vary over M, thus not forming a vector bundle. An explicit example was given in Lecture 4.

Let's examin this closer:

Let $M \subseteq \mathbb{C}^n$, codim $= d$, be defined by $\rho_1 = \dots = \rho_d = 0$, $d\rho_1 \wedge \dots \wedge d\rho_d \neq 0$.

Recall $d\rho = \partial\rho + \bar\partial\rho = 2\operatorname{Re}\partial\rho$,

and $\partial\rho = \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j} dz_j$

How can the $d\rho_k$ be linearly independent while the $\partial\rho_k$ are not?

Answer: linear (in)dependence of $\partial\rho_k$ is over $\mathbb{C}$!

Ex 1. $\partial\rho_1 = i\,\partial\rho_2$. Then $\partial\rho_1, \partial\rho_2$ are <u>not</u> linearly independent / $\mathbb{C}$.

Let us write $\partial\rho_2 = \alpha + i\beta$, where α, β are real 1-forms. Then, $\partial\rho_1 = i(\alpha + i\beta) = -\beta + i\alpha$. Now, $d\rho = 2\operatorname{Re}\partial\rho$, so $d\rho_1$ and $d\rho_2$ linearly indepent means α, β linearly independent / $\mathbb{R}$.

Def. A real submfld $M \subseteq \mathbb{C}^n$ of codim $= d$ is called *generic* if its defining functions $\rho_1 = \dots = \rho_d = 0$ satisfy
$\partial\rho_1 \wedge \dots \wedge \partial\rho_d \neq 0$.

Rem. A generic submanifold M is CR, with CR dim $= \dim_{\mathbb{C}} T_p^{1,0} M = n-d$.

For completeness, we record also the following for arbitrary real submanifolds (not nec. CR). The proof, left as an exercise is based on Ex 1.

Prop. Let $M \subseteq \mathbb{C}^n$ be of codim $= d$.

① If d is even then $\dim T_p^{1,0} M$ can vary in the following range

$$n-d \leq \dim T_p^{1,0} M \leq n - \frac{d}{2}$$

② If d is odd,

$$n-d \leq T_p^{w_0} M \leq n - \frac{d+1}{2}.$$